Map-functor Factorized

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February 18, 1991

It is well known that any initial data type comes equipped with a so-called map-functor. We show that any such map-functor is the composition of two functors, one of which is —closely related to— the data type functor, and the other is —closely related to— the function μ (that for any functor F yields an initial F-algebra, if it exists).

Notation

Let K be a category, and $\mathsf{F} : K \to K$ be an endo-functor on K. Then $\mu\mathsf{F}$ denotes "the" initial F-algebra over K, if it exists. Further, $\mathcal{F}(K)$ is the category of endo-functors on K whose morphisms are, as usual, natural transformations; and $\mathcal{F}_\mu(K)$ denotes the full sub-category of $\mathcal{F}(K)$ whose objects are those functors F for which $\mu\mathsf{F}$ exists.

For mono-functors F, G and bi-functor $\dagger$ we define the composition FG by $x(\mathsf{FG}) = (x\mathsf{F})\mathsf{G}$, and we denote by $\mathsf{F} \dagger \mathsf{G}$ the mono-functor defined by $x(\mathsf{F} \dagger \mathsf{G}) = x\mathsf{F} \dagger x\mathsf{G}$. Object A when used as a functor is defined by $xA = A$ for any object x and $fA = id_A$ for any morphism f. (An alternative notation for $A \dagger \mathsf{I}$ is the 'section' $A\dagger$.) In the examples we assume that $X, \grave{\pi}, \acute{\pi}, \vartriangle$ form a product, and $+, \grave{\iota}, \acute{\iota}, \triangledown$ a co-product.

Making μ into a functor

We define a functor $\_^\mu : \mathcal{F}_\mu(K) \to K$ that is closely related to μ, and has therefore a closely related notation. For any $\mathsf{F}, \mathsf{G} \in Obj(\mathcal{F}_\mu(K))$ and $\phi : \mathsf{F} \dot{\to} \mathsf{G}$ we put

(1) $\mathsf{F}^\mu \;=\;$ target of $\mu\mathsf{F}$

(2) $\phi^\mu \;=\; (\!|\mathsf{F}|\, \phi ; \mu\mathsf{G}|\!) \;:\; \mathsf{F}^\mu \to \mathsf{G}^\mu.$

Notice that by (1) we have $\mu\mathsf{F} : \mathsf{F}^\mu\mathsf{F} \to \mathsf{F}^\mu$. (Some authors in the Squiggol community are used to define $(L, in) = (\mathsf{F}^\mu, \mu\mathsf{F})$.) The instance of ϕ that has to be taken in the right-hand side of (2) is $\phi_{\mathsf{G}^\mu} : \mathsf{G}^\mu\mathsf{F} \to \mathsf{G}^\mu\mathsf{G}$; the typing $\phi^\mu : \mathsf{F}^\mu \to \mathsf{G}^\mu$ is then easily verified. In order to prove that $\_^\mu$ satisfies the two other functor axioms, we present a lemma first.

(3) Lemma *For $\phi : \mathsf{F} \dot{\to} \mathsf{G}$ and $\psi : A\mathsf{G} \to A$,*

$$(\!|\mathsf{F}|\, \phi ; \psi|\!) \;=\; (\!|\phi ; \mu\mathsf{G}|\!) ; (\!|\mathsf{G}|\, \psi|\!).$$

Proof (Within this proof we use the law names and notation of Fokkinga & Meijer [1]. The reader may easily verify the steps by unfolding $f : \phi \xrightarrow{\mathsf{F}} \psi$ into $\phi ; f = f\mathsf{F} ; \psi$, and using $\phi : \mathsf{F} \dot{\to} \mathsf{G} \equiv (\forall f :: f\mathsf{F} ; \phi = \phi ; f\mathsf{G})$.)

$$
\begin{array}{ll}
 & \text{required equality} \\
\Leftarrow & \quad \textsc{Fusion} \\
 & (\!|\mathsf{G}|\ \psi|\!) : \phi ; \mu\mathsf{G} \xrightarrow{\mathsf{F}} \phi ; \psi \\
\Leftarrow & \quad \textsc{Ntrf to Homo}, \phi : \mathsf{F} \dot{\to} \mathsf{G} \\
 & (\!|\mathsf{G}|\ \psi|\!) : \mu\mathsf{G} \xrightarrow{\mathsf{G}} \psi \\
\equiv & \quad \textsc{Cata Homo} \\
 & \textbf{true}.
\end{array}
$$

(End of proof)

It is now immediate that $\_^{\mu}$ distributes over composition. For $\phi : \mathsf{F} \dot{\to} \mathsf{G}$ and $\psi : \mathsf{G} \dot{\to} \mathsf{H}$ we have $\phi ; \psi : \mathsf{F} \dot{\to} \mathsf{H}$ and

$$
\begin{array}{ll}
 & (\phi ; \psi)^{\mu} \\
= & \\
 & (\!|\mathsf{F}|\ \phi ; \psi ; \mu\mathsf{H}|\!) \\
= & \quad \text{Lemma (3), noting that } \psi ; \mu\mathsf{H} : \mathsf{H}^{\mu}\mathsf{G} \to \mathsf{H}^{\mu} \\
 & (\!|\mathsf{F}|\ \phi ; \mu\mathsf{G}|\!) ; (\!|\mathsf{G}|\ \psi ; \mu\mathsf{H}|\!) \\
= & \\
 & \phi^{\mu} ; \psi^{\mu}.
\end{array}
$$

It is also clear that $id^{\mu} = id$. Thus, $\_^{\mu}$ is a functor, $\_^{\mu} : \mathcal{F}_{\mu}(K) \to K$.

(4) Remark Another corollary of the lemma is this: for $\phi : \mathsf{F} \dot{\to} \mathsf{G}$ we have that $\phi^{\mu} ; f$ is a catamorphism whenever f is a catamorphism. (The typing determines that the former is an F-catamorphism, and the latter a G-catamorphism.) □

Let us look at some $\phi : \mathsf{F} \dot{\to} \mathsf{G}$ and see what ϕ^{μ} is.

Example Probably the most simple, non-trivial, choice is $\mathsf{F}, \mathsf{G} := \mathbf{1} + A X \mathsf{I},\ \ \mathbf{1} + \mathsf{I}$ and $\phi := id + \acute{\pi}$. Notice that F^{μ} = the (set L of) cons-lists and $\mu\mathsf{F} = nil \triangledown cons$, G^{μ} = the (set $\mathbb{N}$ of) naturals and $\mu\mathsf{G} = zero \triangledown suc$. We find

$$\phi^{\mu} \quad = \quad (\!|\mathsf{F}|\ id + \acute{\pi} ; zero \triangledown suc|\!) \quad = \quad size \quad : \quad L \to \mathbb{N}.$$

□

Example Another non-trivial choice is $\mathsf{F} = \mathsf{G} = A + \mathbb{I}$, so that $\mathsf{F}^{\mu} = \mathsf{G}^{\mu}$ = the (set of) non-empty binary join trees over A, and $\mu\mathsf{F} = tip \triangledown join$. Apart from the trivial $id : \mathsf{F} \dot{\to} \mathsf{G}$, we have $\phi := id + \bowtie : \mathsf{F} \dot{\to} \mathsf{G}$ where $\bowtie = \acute{\pi} \vartriangle \grave{\pi}$. We have

$$\phi^{\mu} \quad = \quad (\!|id + \bowtie ; tip \triangledown join|\!) \quad = \quad \bowtie / \quad = \quad reverse.$$

Since $\_^{\mu}$ is a functor, we have a simple proof that *reverse* is its own inverse:

$$
\begin{aligned}
& \quad reverse ; reverse \\
= & \quad \phi^{\mu} ; \phi^{\mu} \\
= & \qquad \text{functor axiom} \\
& \quad (\phi ; \phi)^{\mu} \\
= & \qquad \text{easy: } \bowtie ; \bowtie = id \\
& \quad id^{\mu} \\
= & \quad id .
\end{aligned}
$$

Notice also that by Remark (4), $reverse ; f$ is a catamorphism whenever f is. □

Example Let $\dagger$ be a bi-functor and let $\mathrm{F} = A \dagger \mathrm{I}$ and $\mathrm{G} = \mathbf{1} \dagger \mathrm{I}$. Take $\phi = ! \dagger id : A \dagger \mathrm{I} \dot{\to} \mathbf{1} \dagger \mathrm{I}$. Then

$$\phi^{\mu} \;=\; (\!| A \dagger \mathrm{I} |\; ! \dagger id ; \mu(\mathbf{1} \dagger \mathrm{I}) |\!) \;=\; shape \;\; (= \;\; !\text{-map}).$$

□

Factorizing map-functors

Let $\dagger$ be any bi-functor for which $\mu(A \dagger \mathrm{I})$ exists for all A. Recall that *the map-functor induced by* $\dagger$, $\_^{\varpi}$ say, is defined by

$$
\begin{aligned}
A^{\varpi} &= \text{target of } \mu(A \dagger \mathrm{I}) \\
f^{\varpi} &= (\!| A \dagger \mathrm{I} |\;\; f \dagger id ; \mu(B \dagger \mathrm{I}) |\!) \;\; : \;\; A^{\varpi} \to B^{\varpi}
\end{aligned}
$$

for $f : A \to B$. We shall now define a functor $\_^{\dagger} : K \to \mathcal{F}_{\mu}(K)$ in such a way that composed with $\_^{\mu} : \mathcal{F}_{\mu}(K) \to K$ it equals the map-functor $\_^{\varpi} : K \to K$. To this end define

$$
\begin{aligned}
A^{\dagger} &= A \dagger \mathrm{I} \\
f^{\dagger} &= f \dagger id \;\; : \;\; A \dagger \mathrm{I} \dot{\to} B \dagger \mathrm{I} \qquad (\text{with } (f \dagger id)_C = f \dagger id_C)
\end{aligned}
$$

for any $f : A \to B$. (That $f^{\dagger}$ is a natural transformation is easily verified; it also follows from laws NTRF TRIV, NTRF ID, NTRF BI-DISTR from Fokkinga & Meijer [1].) Indeed

$$
\begin{aligned}
A^{\dagger \mu} &= (A \dagger \mathrm{I})^{\mu} \;=\; A^{\varpi} \\
f^{\dagger \mu} &= (f \dagger id)^{\mu} \;=\; (\!| A \dagger \mathrm{I} |\;\; f \dagger id ; \mu(B \dagger \mathrm{I}) |\!) \;=\; f^{\varpi} .
\end{aligned}
$$

So $\varpi = {}^{\dagger}\mu$.

Remark It can be shown that $\_^{\dagger}$ is just $curry(\dagger)$. (Here $curry(\_)$ is the well-defined functor from the category $\mathbf{A} \times \mathbf{B} \to \mathbf{C}$ to the category $\mathbf{A} \to (\mathbf{B} \to \mathbf{C})$, where each arrow denotes a category of functors with natural transformations as morphisms.) Thus, given bi-functor $\dagger$, we can express its map-functor without further auxiliary definitions as $curry(\dagger)$ composed with $^{\mu}$. □

References

[1] M.M. Fokkinga and E. Meijer. Program calculation properties of continuous algebras. December 1990. CWI, Amsterdam.