Reducing hopeful majority

Lambert Meertens

We are given a non-empty bag of (votes on) 'candidates', and are asked to determine if some candidate has the majority.

Several derivations of linear-time algorithms have been given, all of which work in two phases: first find a 'hopeful majority' candidate, and next check if it really has the majority. A 'hopeful majority' candidate is any candidate c in the bag such that if some candidate has the majority, it is c.

I consider here only the problem of finding some hopeful majority candidate. All previous algorithms I have seen basically scan the bag. The purpose of this note is to show that there is a divide-and-rule approach. In a previous note I have given a derivation, mainly based on predicate calculus. Here only the solution is presented.

Let C stand for the type of the candidates, and N for the naturals. The operation $\oplus$: $(C \times N) \times (C \times N) \rightarrow C \times N$ is defined by:

$$(c0, d0) \oplus (c1, d1) \quad = \quad (c0, d0+d1) \prec c0 = c1 \succ ((c0, d0-d1) \uparrow_{\pi_2} (c1, d1-d0)) \quad ,$$

where $a \prec p \succ b$ stands for **if** p **then** a **else** b **fi**. We also define f: $C \rightarrow C \times N$:

$$f \bullet c \quad = \quad (c, 1) \quad .$$

Now a hopeful majority candidate is determined by $\pi_2 \bullet h$, where h is the bag homomorphism defined by

$$h \quad = \quad \oplus / \bullet f* \quad .$$

However, something funny is going on here. Even under the hot indeterminate interpretation of $\uparrow_{\pi_2}$ the operator $\oplus$ is not associative. This can be seen by considering the different ways to compute h on a bag of three distinct candidates. According to the current definitions this would mean that h is not a proper bag homomorphism. Associativity is not required for consistency if we consider the bag splitting itself also as indeterminate. This is already mentioned in the *Algorithmics* paper, but I had not come across a clear (non-contrived) example before. The definition of the indeterminate bag reduce $\oplus/$ is then that it is the 'thinnest' (most determinate) indeterminate function r satisfying

$$r \bullet \tau \rightsquigarrow c \quad ;$$