The continuum as a final coalgebra

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Abstract

We define the continuum up to order isomorphism, and hence up to homeomorphism via the order topology, in terms of the final coalgebra of either the functor $N \times X$, product with the set of natural numbers, or the functor $1 + N \times X$. This makes an attractive analogy with the definition of N itself as the initial algebra of the functor $1 + X$, disjoint union with a singleton. We similarly specify Baire space and Cantor space in terms of these final coalgebras. We identify two variants of this approach, a coinductive definition based on final coalgebraic structure in the category of sets, and a direct definition as a final coalgebra in the category of posets. We conclude with some paradoxical discrepancies between continuity and constructiveness in this setting. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The algebra $(N, 0, s)$ of non-negative integers under successor s with constant 0, the basic number system of mathematics, occupies a distinguished position in the category of algebras having a unary operation and a constant, namely as its initial object.

In this paper we show that the continuum, the basic environment of physics, occupies an equally distinguished position in the category of coalgebras having one "dual operation" of arity N, namely as its final object. Here a dual operation is understood as a function $X \to X + X + \cdots$ dual to an operation as a function $X \times X \times \cdots X$, with arity being the number of copies of X in each case.

Coinduction as the engine of final coalgebras has only recently relatively been recognized as a genuine logical principle [2], sibling to induction as the engine of initial algebras. Before that, it was introduced and used mostly in the semantics of concurrency [18]. It has by now been presented from many different angles: [1, 10, 16, 21–23], to name just a few contributors.

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Why would so foundational a principle wait for the late 20th century to be discovered? In [19, 21] the idea was put forward that coinduction is new only by name, while it had actually been around for a long time, concealed within the infinitistic methods of *mathematical analysis*. Roughly, the idea is that

$$\frac{\text{induction}}{\text{arithmetic}} \approx \frac{\text{coinduction}}{\text{analysis}}.$$

The basic passage to infinity in elementary calculus is coinductive, dual to the inductive passage to infinity in elementary arithmetic. However, all evidence presented in [21] built on a datatype of real numbers, assumed given. In the present note, we describe several ways to derive this datatype from first principles, as a final coalgebra.

In its simplest and most abstract form, the idea is to view an infinite object x as a stream, i.e. an infinite list $[x_0, x_1, x_2 \ldots]$ of elements from some set Σ. Construed as a rudimentary process, this stream reduces to an *action* or "head"

$$h(x) = x_0$$

and a *continuation* or "tail", another stream

$$t(x) = x' = [x_1, x_2, x_3 \ldots].$$

But the pair

$$\langle h, t \rangle : \Sigma^N \to \Sigma \times \Sigma^N$$

is, of course, the final coalgebra structure for the functor $\Sigma \times (-): \mathsf{Set} \to \mathsf{Set}$.

Although this simple picture, and the related coalgebraic ideas, are nowadays probably not far from becoming a standard part of the basic toolkit for designing datatypes, and perhaps even systems in general [10, 23], it may nevertheless come as a surprise that the idea to implement real numbers along these lines can be dated as far back as 1971 – and to the writings of the first "hackers", of all places! In the famous Hakmem report, R.W. Gosper and R. Schroeppel took up analyzing (among some 200 other computational themes) real arithmetic in terms of continued fractions. The main result appears to be Gosper's derivation [8, 101B] of a general algorithmic scheme for implementing arithmetic operations and methods of successive approximation.

He begins as follows:

Let x be a continued fraction

$$p_0 + \cfrac{q_0}{p_1 + \cfrac{q_1}{p_2 + \cdots}} = p_0 + \frac{q_0}{x'},$$

where x' is again a continued fraction and the p's and q's are integers.[…]

Instead of a list of p's and q's, let x be a computer *subroutine* which produces its next p and q each time it is called. Thus on its first usage, x will "output" p_0 and q_0 and, in effect, change itself into x'.

Real numbers are thus presented as streams of pairs of integers. The mentioned coalgebraic structure, although never spelled out, is then employed in the subsequent

constructions, as well as in the discussion touching upon some of the still very active themes, such as guarded induction [19, 20], or the role of redundancy in representation.

Although Hakmem was never published, it has remained available throughout the intervening years, it was widely read, and sometimes it is even cited. But while the algorithms derived in it have been acknowledged as the source of inspiration for some of the most interesting modern approaches to exact real arithmetic [7, 14, 11, 25], the underlying coalgebraic idea seems to have gone unnoticed.

While hoping to point to this conceptual link, we must add that the constructions on the following pages should not be taken as a rational reconstruction of the Hakmem view of reals. In fact, they were obtained while we were trying to work out an effective underpinning for our wider calculus-by-coinduction effort, initiated in [19, 21]. The coalgebras presented here are just one detail in that plan. However, the realization that a concrete coalgebra of reals was never worked out, although its effects were in use for quite a while, made us try to present it here.

2. The basic result

The core of our main result can be found in the following variant of a bijection due to Hausdorff [9, Section 10] between sequences of non-negative integers and reals in $[0, 1)$. Consider the sequence $0, 3, 1, 0, 4, 0, 0, 0 \ldots$ of non-negative integers. If we write the integers in unary or tally notation (so 3 becomes a block of three 1's) and the commas as 0's, and put a point at the start, we obtain $0.011101001111000 \ldots$, which is the binary expansion of a real in the half-open unit interval $[0, 1)$, namely (in decimal) $0.456787109375 \ldots$.

Since infinite binary expansions have infinitely many 0's and no infinite run of 1's, this translation is reversible: given the binary expansion of any real in $[0, 1)$, read its 0's as commas and its runs of 1's between sucessive 0's as non-negative integers in tally notation, with the first integer in the sequence being the number of 1's preceding the first 0. We therefore have a one–one correspondence or bijection between N^N, the set of infinite sequences of non-negative integers, and $[0, 1)$, the unit interval of reals.

Under the lexicographic ordering of N^N (Hausdorff's variant used alternating lexicographic order in conjunction with a different coding) and the standard ordering of $[0, 1)$, this bijection is monotone. This is because at the first integer where the sequences differ, the sequence with the smaller integer corresponds to a real with a shorter run of 1's, which is therefore numerically smaller. Hence N^N lexicographically ordered has the order type of the continuum $[0, 1)$. The continuum is standardly topologized by its order topology, namely that generated by its open intervals, whence N^N under this topology is homeomorphic to the continuum.

Readers familiar with coalgebras will recognize N^N as the canonical final coalgebra of sort $X \to N \times X$. The above bijection is made an isomorphism of coalgebras by furnishing $[0, 1)$ with coalgebraic structure corresponding to that of N^N. Doing so makes $[0, 1)$ a final coalgebra of the same sort, and the claim in our title is now demonstrated.

This is by no means the full story however. First, there is no canonical choice of bijection. Putting the above bijection on hold for the time being, we consider a number of other equally good bijections, in particular some that connect up with continued fractions.

Second, we can obtain a stronger result, namely for coalgebras whose carrier is a poset, where we are able to specify the order type of the continuum directly as the final coalgebra for a suitable functor (Theorem 5.1). However algebraic and coalgebraic structure is more usually imposed on sets, and the final-coalgebraic structure of the continuum can already be specified, albeit not so directly, in this traditional setting.

3. Coalgebras on sets

We will start with the even more traditional decomposition of algebras into operations, which we mimic for coalgebras, deferring for the moment the more general functorial treatment. For any set A, an *operation* of arity A is a function $X^A \to X$. And for any set A, a *dual operation* of arity A is a function $X \to A \times X$,[1] equivalently a pair of functions $h : X \to A$, $t : X \to X$ sometimes called *destructors* (so with this terminology it takes two destructors to make one dual operation). In this narrow sense, an *algebra* on X is a family f_i of operations on X, while dually a *coalgebra* on X is a family f_i of dual operations on X.

A *homomorphism of algebras* X, Y is a function $f : X \to Y$ such that for every operation $g : X^A \to X$ with corresponding operation $g' : Y^A \to Y$, $f(g(x_0, x_1, \ldots)) = g'(f(x_0), f(x_1), \ldots)$. Hence any family A_i of arities, or *signature*, determines the category of all algebras with that signature.

Dually a *homomorphism of coalgebras* X, Y is a function $f : X \to Y$ such that for every dual operation $g = (h, t) : X \to A \times X$ with corresponding $g' = (h', t') : Y \to A \times Y$, $h(x) = h'(f(x))$ and $f(t(x)) = t'(f(x))$. This similarly defines the category of all coalgebras with a given signature.

Although the homomorphism conditions for coalgebras are not obviously dual to those for algebras, each condition can be expressed in an evident way as commutativity of one of the following diagrams for, algebras and coalgebras, respectively, which brings out the duality better.

$$
\begin{array}{cccc}
X \times X \times \cdots \xrightarrow{g} X & \quad & X \xrightarrow{g} X + X + \cdots \\
f \times f \cdots \downarrow \qquad \downarrow f & \quad & f \downarrow \qquad \downarrow f + f \cdots \\
Y \times Y \times \cdots \xrightarrow{g'} Y & \quad & Y \xrightarrow{g'} Y + Y + \cdots
\end{array}
$$

The algebra $(N, 0, s)$ is well-known to be initial in the category of algebras with signature $(0, 1)$ (one constant and one unary operation). And the coalgebra (A^N, h, t) of all infinite sequences $f : N \to A$, where $h(f) = f(0)$ (head) and $t(f) = f \circ \sigma$ (tail),

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[1] Just as X^A denotes the product $X \times \cdots \times X$ of A copies of X, so dually does $A \times X$ denote the sum or disjoint union $X + X + \cdots$ of an A-indexed family of copies of X.

is well-known to be final in the category of coalgebras with signature (A) (organizing $(h, t) : X \to A \times X$ as a single A-ary dual operation) [18].

In the following we write $\mathbb{I} = [0, 1)$ for the half-open unit interval of reals, and $\mathbb{R}^+$ for the non-negative reals, and jointly define the functions $\lfloor x \rfloor$ and $x \bmod 1$, of respective types $\mathbb{R} \to \mathbb{Z}$ and $\mathbb{R} \to \mathbb{I}$, via $\lfloor x \rfloor + (x \bmod 1) = x$.

Theorem 3.1. *The continued fraction coalgebra $\mathscr{A} = (\mathbb{I}, H, T)$, the continuum from 0 to 1, is final in the category of coalgebras with signature (N), where $H : \mathbb{I} \to N$ satisfies $H(x) = \lfloor x/(1 - x) \rfloor$ and $T : \mathbb{I} \to \mathbb{I}$ satisfies $T(x) = x/(1 - x) \bmod 1$.*

Proof. Here H and T pair up as a function $(H, T) : \mathbb{I} \to N \times \mathbb{I}$, i.e. a dual operation of arity N. It will be helpful in following the proof to understand the nature of (H, T), which if $\mathscr{A}$ is a final coalgebra must be an isomorphism [15]. This (H, T) can be characterized as the composite

$$[0, 1) \cong [0, \infty) = [0, 1) \cup [1, 2) \cup \cdots \cong [0, 1) + [0, 1) + \cdots,$$

where the first isomorphism is given by $x/(1 - x)$, the second is identity, and the third puts each unit interval $[i, i + 1)$ in monotone bijection with the corresponding copy of $\mathbb{I}$.

Since $H(x) + T(x) = x/(1 - x)$, we obtain $x = 1 - 1/(1 + H(x) + T(x))$, constituting the coinductive step of this form of continued fraction. Taking two more steps yields $x = 1 - 1/(1 + H(x) + 1 - 1/(1 + HT(x) + 1 - 1/(1 + HTT(x) + TTT(x))))$, indicating that in the limit $x = 1 - 1/(2 + A_0 - 1/(2 + A_1 - 1/\ldots$ where $A_i = HT^i(x)$.

A theorem underpinning continued fraction theory is that there exists exactly one non-negative value for continued fractions of this form when the A_i's are non-negative.[2] By an easy induction this value must lie in $\mathbb{I}$.

Since $\Omega = (N^N, h, t)$ is the final coalgebra with signature (N), there exists a unique homomorphism $f : \mathscr{A} \to \Omega$, namely that mapping x to its associated continued fraction sequence $(A_0 = H(x), \ A_1 = HT(x), \ A_2 = HTT(x), \ldots)$.[3] Since x may be recovered from this sequence as the unique value of the continued fraction, f is injective. And since every sequence $(A_0, A_1, \ldots)$ in Ω evaluates as a continued fraction to a value in $\mathbb{I}$, f is surjective. Hence $f : \mathscr{A} \to \Omega$ is a bijection, and therefore an isomorphism of coalgebras,[4] making $\mathscr{A}$ also final. $\square$

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[2] Consider the sequence $1, 1, 1, \ldots$, whose two possible values are the roots of $x^2 + x - 1$. The proscription on negative values rules out the negative root.

[3] The reader who finds it obvious that any coalgebra with this signature has a unique homomorphism to Ω is observing the obviousness of the finality of Ω.

[4] That bijective homomorphisms are isomorphisms is proved most simply for the more general definition of coalgebra we will be using shortly, a morphism $X \to F(X)$. If the homomorphism $f : X \to Y$ is a bijection, more generally an isomorpism of the underlying-object category, then so is $F(f)$ whence we have not only $f^{-1} : Y \to X$ but also $F(f)^{-1} : F(Y) \to F(X)$. But $F(f)^{-1} = F(f^{-1})$ making this pair of inverses itself a coalgebra homomorphism. The same argument applies to homomorphisms of algebras $F(X) \to X$.

We have now seen two final coalgebras for the signature having one N-ary dual operation. Being final they are necessarily isomorphic coalgebras. Furthermore the unique isomorphism between them is monotone so they are order isomorphic. The question then naturally arises for such coalgebras, does finality determine order type?

Among the topological siblings of the continuum are Baire space and Cantor space. We view these spaces as follows. Just as $\mathbb{I}$ represents the continuum with a least but no greatest element (by no means uniquely, witness the homeomorphic representative $[0, \infty)$), so does the set $\underline{(0,1)} = (0,1) - \mathbb{Q}$ of irrationals between 0 and 1 represent Baire space, while the set $\overline{[0,1]} = [0,1] + (\mathbb{Q} \cap (0,1))$ of reals between 0 and 1 inclusive with every rational in its interior duplicated represents Cantor space.

For all three of $\mathbb{I}$, $\underline{(0,1)}$, and $\overline{[0,1]}$, the topological space so represented has the order interval topology, namely that generated by the open intervals $(x,z) = \{y \mid x < y < z\}$ as basic open sets. The order determining this topology is the standard order, with the additional requirement in the case of Cantor space that duplicate rationals are linearly ordered (for definiteness suppose that the instance of m/n coming from $[0, 1]$ is greater than the instance of m/n coming from $\mathbb{Q} \cap (0,1)$), which ensures that $\overline{[0,1]}$ is ordered linearly like its siblings. [5]

Finality alone of course cannot determine the order type of the continuum, only its cardinality. However one might hope to be able to read off its topology in some natural way from the final coalgebra for this signature. We dash this hope by associating two equally reasonable topologies to the final coalgebra.

Theorem 3.2. *The continued fraction coalgebra $\mathscr{B} = (\underline{(0,1)}, \underline{H}, \underline{T})$, Baire space, is final in the category of coalgebras with signature (N), where $\underline{H} : \underline{(0,1)} \to N$ satisfies $\underline{H}(x) = \lfloor (1-x)/x \rfloor$ $(= \lfloor 1/x \rfloor - 1)$ and $\underline{T} : \underline{(0,1)} \to \underline{(0,1)}$ satisfies $\underline{T}(x) = (1-x)/x \bmod 1$ $(= 1/x \bmod 1)$.*

Proof. In this coalgebra, the isomorphism $(\underline{H}, \underline{T})$ we aim to prove final is given by the composite

$$\underline{(0,1)} \cong \underline{(0,\infty)} = \underline{(0,1)} \cup \underline{(1,2)} \cup \cdots \cong \underline{(0,1)} + \underline{(0,1)} + \cdots,$$

where the underlining indicates absence of rationals, the first isomorphism is given by $(1-x)/x$, the second is identity, and the third puts every unit interval in antimonotone bijection with $\underline{(0,1)}$.

Here $\underline{H}(x) + \underline{T}(x) = (1-x)/x$, so $x = 1/(1 + \underline{H}(x) + \underline{T}(x))$, making the continued fraction associated with $\mathscr{B}x = 1/(1 + A_0 + 1/(1 + A_1 + 1/\ldots$. This type of continued fraction has the same property as the continued fraction associated to $\mathscr{A}$: there exists a unique non-negative value for it when the A_i's are non-negative. In fact it

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[5] Emphasizing the non-uniqueness of these representations, it is a nice exercise to verify that this representation of Cantor space is homeomorphic to the more commonly given representation, which recursively removes an open interval from the middle third of each interval of a growing family of closed intervals starting with $[0,1]$.

is the more commonly used form of continued fraction in practice, customarily being presented as $/A_0 + 1, A_1 + 1, \ldots /$ (whence no zeroes in the sequence), which we would have had here too had we simply taken $N = \{1, 2, \ldots\}$.

That this algebra is well-defined follows from the fact that $\underline{T}$ preserves irrationals. We now show as in the previous theorem that the unique $f : \mathscr{B} \to \Omega$ is an isomorphism. It is injective for the same reason as before, and is surjective because every infinite continued fraction of this kind evaluates to an irrational. (This is well known but will become apparent anyway during the proof of the next theorem.)   □

We now have two non-homeomorphic final coalgebras for the same signature. Neither stands out as more natural than the other, dashing any hope of associating a topology to this signature by consideration of say the most natural final coalgebraic structure for this signature.

Since the kind of continued fraction associated to coalgebra $\mathscr{A}$ is less commonly used than that associated to $\mathscr{B}$ [12, Section 4.5.3], it is natural to ask whether the latter can do any better. One might suppose that doing so would be just a matter of completing $(0, 1)$ to $[0, 1)$ in $\mathscr{B}$, but there is one seemingly insignificant problem: $(1 - x)/x$ is not defined at zero. Omitting only zero from $[0, 1)$ does not solve this because $\underline{T}$ sends some non-zero values to zero, and in fact every rational is sent to zero by some $\underline{T}^i$.

Our solution is to change signatures. For this however we must pass from our preliminary definition of coalgebra as a family of dual operations to the more general notion of a coalgebra as a function $X \to F(X)$ where $F : \mathsf{Set} \to \mathsf{Set}$ is an arbitrary functor on the category of sets. With the preliminary notion our functors were of the form $\Pi_i(A_i \times X)$ where the A_i's are the arities of the signature. The new functors we admit include $1 + A \times X = 1 + X + X + \cdots$, where we take 1 to be the singleton $\{\bigstar\}$, this being the functor we want (with $A = N$) for the following theorem.

It is convenient here to represent a function $f : X \to 1 + A \times X$ as a pair of functions $H, T : X' \to A \times X$, head and tail. The common domain $X' \subseteq X$ of H and T consists of those $x \in X$ such that $f(x)$ lands in $A \times X$, and $(H(x), T(x)) = f(x)$. For those x such that $f(x)$ lands in $1 = \{\bigstar\}$, we adopt the convention that $(H, T)(x) = \bigstar$.

The canonical final coalgebra for the functor $1 + A \times X$ is well known to have carrier $A^* \cup A^N$, namely all finite and infinite sequences over A. When $A = N$ we denote this final coalgebra by Ω'. Head and tail work as for Ω but with both head and tail of the empty sequence ε being undefined, i.e. $(H, T)(\varepsilon) = \bigstar$.

The coalgebra of the following theorem is the appropriate one for the commonest notion of a continued fraction $/A_0, A_1, \ldots /$ as a representation of the real $1/(A_0 + 1/(A_1 + \cdots))$ in $\mathbb{I}$, for *positive* integers $A_i \geqslant 1$ [12, Section 4.5.3]. We depart from this representation in two respects. First, Knuth takes the value of a finite sequence $A_0, \ldots, A_n$, to be $1/(A_0 + 1/(A_1 + \cdots + 1/A_n)\ldots))$, but as he points out this identifies the values of $(A_0, \ldots, A_n + 1)$ and $(A_0, \ldots, A_n, 1)$, destroying bijectivity of the correspondence between $(0, 1]$ and the set of finite and infinite sequences of positive integers. We repair this by taking the value of a finite sequence to be instead

$1/(A_0 + 1/(A_1 + \cdots + 1/(A_n + 1)\ldots))$, in effect truncating the infinite sequence just before the division sign instead of just before the addition sign. Second, as a matter of esthetics we use sequences of *non-negative* integers, achieved simply by subtracting 1 from all elements of a sequence. To obtain the value of such a translated sequence we must add the 1's back, as in $1/(1 + A_0 + 1/(1 + A_1 + \cdots + 1/(1 + A_n + 1)\ldots))$ for infinite sequences and $1/(1 + A_0 + 1/(1 + A_1 + \cdots + 1/(1 + A_n + 1)\ldots))$ for finite.

Theorem 3.3. *The continued fraction coalgebra* $\mathscr{A}' = ((0,1], H', T')$, *defined by* $H'(x) = \lceil 1/x \rceil - 2$ *where* $H' : (0,1) \to N$ *and* $T'(x) = 1/x - H'(x) - 1 = 1 - (-1/x \bmod 1)$ *where* $T' : (0,1) \to (0,1]$, *is final in the category of coalgebras* $X \to 1 + N \times X$.

Proof. The isomorphism (H', T') is given by the composite

$$(0,1] \cong [0,\infty) = \{0\} \cup (0,1] \cup (1,2] \cup \cdots \cong 1 + (0,1] + (0,1] + \cdots,$$

where the first isomorphism is given by $(1 - x)/x$, the second is identity, and the third puts each unit interval $(i, i+1]$ for $i \geqslant 0$ in monotone bijection with the ith copy of $(0,1]$. This is essentially $\mathscr{B}$'s isomorphism with the rationals filled in, which is easier to see when it is noticed that $\lceil 1/x \rceil - 2$ differs from $\lfloor 1 - x/x \rfloor$, and $1 - (-1/x \bmod 1)$ from $1/x \bmod 1$, only when $1/x$ is an integer. The real 1 is omitted from the domain of H' and T' because the first isomorphism pairs it with the element of the singleton 1.

Again we consider the unique homomorphism $\mathscr{A}' \to \Omega'$. On $\underline{(0,1)} \subseteq \mathbb{I}$, since $\underline{H}$ and $\underline{T}$ are the respective restrictions of H' and T' to $\underline{(0,1)}$ we have the same bijection between the set of irrationals $\underline{(0,1)}$ and the set of infinite sequences N^N as in Theorem 3.2. Since every rational can be put in the form of a finite continued fraction of the present kind by Euclid's algorithm, and since every finite continued fraction obviously evaluates to a rational, the rationals of $\mathbb{I}$ are in bijection with the finite sequences of Ω' (and hence only irrationals can map to Ω, explaining the surjectivity in Theorem 3.2). Hence this homomorphism from $\mathscr{A}'$ to Ω' is itself a bijection, and hence an isomorphism.   □

This change of signature has allowed us to represent the whole of the continuum $\mathbb{I}$ as a final coalgebra instead of just the irrationals using standard continued fractions. The omission of the finite sequences from $\mathscr{B}$ might well have been how we managed to sneak Baire space in. However this new signature comes with its own ambiguity: the next theorem exhibits Cantor space as another final coalgebra for the same signature, which like Baire space is not homeomorphic to the continuum.

Theorem 3.4. *The continued fraction coalgebra* $\mathscr{C} = (\overline{[0,1]}, \bar{H}, \bar{T})$, *Cantor space, is final in the category of coalgebras* $X \to N \times X + 1$, *where* $\bar{H} : \overline{[0,1]} - \{1\} \to N$ *satisfies* $\bar{H}(x) = \lfloor x/(1-x) \rfloor - 1$ *and* $\bar{T} : \overline{[0,1]} - \{1\} \to \mathbb{I}$ *satisfies* $\bar{T}(x) = x/(1-x) \bmod 1$ *as a monotone injection* (*thereby preserving both the distinction between, and the order of, duplicate rationals*).

Proof. We are now back to the basic structure of $\mathscr{A}$. The isomorphism $(\bar{H}, \bar{T})$ is given by the composite

$$\overline{[0,1]} \cong \overline{[0,\infty]} = \overline{[0,1]} \cup \overline{[1,2]} \cup \cdots \cup \{\infty\} \cong \overline{[0,1]} + \overline{[0,1]} + \cdots + 1.$$

Here $\overline{[x,y]}$ denotes $[x,y]$ with all rationals in (x,y) duplicated, the first isomorphism is given by $x/(1-x)$, the second is identity, and the third puts every unit interval in monotone bijection with $\mathbb{I}$. Except for the duplicate rationals this is the same picture as for $\mathscr{A}$.

The crucial difference from $\mathscr{A}$ is the treatment of duplication of rationals in the interior of $[0,1]$. The essential accounting takes place at the identity isomorphism: $\overline{[i, i+1]} \cup \overline{[i+1, i+2]}$ duplicates $i+1$, while $\{\infty\}$ is not duplicated because it has no counterpart in the finite intervals. The rest can be reconstructed from the argument for the finality of $\mathscr{A}$. □

4. Pinning down the topology

The previous section showed that each of two functors admitted two structures that, while isomorphic as coalgebras, were neither order isomorphic as posets nor homeomorphic as topological spaces. Thus despite the uniqueness up to isomorphism of final coalgebra structure for a functor, finality does not of itself determine either the order type or the topology of the continuum.

Nevertheless we can specify the order type of the continuum coinductively *in terms of* the final coalgebra $\Omega = (N^N, h, t)$. The situation is analogous to the initial algebra $(N, 0, s)$, whose elements have the form $s^i(0)$, allowing N to be understood unambiguously as the set of natural numbers yet not uniquely determining a monoid structure on N. The standard monoid may however be specified with the inductive definition $x + 0 = x$, $x + s(y) = s(x + y)$. In this case we do not need to say "least partial function" because the definition is primitive recursive and therefore determines a total function.

The coinductive counterpart of this inductive definition of $+$ specifies a unique partial ordering of the continuum via

$$x < y \quad \text{iff} \quad h(x) < h(y) \text{ or } (h(x) = h(y) \text{ and } t(x) < t(y)),$$

where x, y are distinct sequences (when they are the same we have $x \leqslant y$ by reflexivity). This is the coinductive definition of lexicographic ordering, under the assumption that N is standardly ordered. Continuing the analogy with $+$, this coinductive definition of $\leqslant$ does not require "greatest partial order" because the definition alone already determines a total order, and therefore is uniquely determined. (Any two distinct sequences have the form $x@y$, $x@z$ where $@$ denotes concatenation (LISP *append*), x is their longest common prefix, and either $h(y) < h(z)$ or $h(z) < h(y)$.)

Without changing the coalgebraic structure on N^N, we can also specify the order type of Baire space coinductively, and uniquely by the same reasoning, as follows.

$$x \leqslant y \quad \text{iff} \quad h(x) < h(y) \text{ or } (h(x) = h(y) \text{ and } t(x) \geqslant t(y)).$$

This is the definition of alternating lexicographic order on N^N, being the order on the coalgebra of Theorem 3.2.

For $N^* \cup N^N$, these two coinductive definitions may be used without change, yielding the order types of respectively Cantor space and the continuum, being the orders on the coalgebras of Theorems 3.4 and 3.3, respectively.

Remark. What exactly the two definitions displayed above coinductive? From the outset, coinduction was, of course, specified as the proof principle supported by the final coalgebras, just like induction is the proof principle supported by the initial algebras. However, while the familiar inductive schemes, interpreted in categories, boil down to constructing homomorphisms from initial algebras, constructing homomorphisms to final coalgebras covers only a small part of what coinduction should clearly capture. For instance, only countably many reals can be captured as constant anamorphisms into any of our coalgebras of reals. On the syntactic level, induction is easily recognized by its base-case-step-case beat, whereas coinduction is either intuitively described as "induction without a base case", or taken to denote some property of final coalgebras.

In any case, the above coinductive definitions of $<$ and $\leqslant$ can be viewed as guarded equations in the form

$$< \; = \; \gamma(<),$$

where the operation γ is guarded by the occurrences of the destructor h. In general, guarded induction over streams usually boils down to induction, with the equations determining the head as the base case.

5. Coalgebras on posets

Returning to the analogy with natural numbers, there is a more direct way of specifying the monoid structure on the natural numbers, namely as the free monoid on one generator. This may be formulated as an initial algebra by expanding the signature for monoids with a constant, with no additional laws governing that constant. The monoid of natural numbers is then recovered as the monoid of the initial algebra among those algebras of type (A, f, c, d) (i.e. with signature $2-0-0$, meaning a binary operation and two constants) such that (A, f, c) is a monoid, i.e. f is associative and c is both the left and right identity element for f.

This prompts the question of whether the order type of the continuum can be obtained directly as a final coalgebra by working in the category of posets.

The different topologies we have been able to assign to the final coalgebra of each of the functors $X \to N \times X$ and $X \to 1 + N \times X$ required for each functor two ostensibly

different isomorphisms (H, T), structurally speaking. We get the same final coalgebra because we cannot encode the difference between the isomorphisms in terms of functors, the differences being merely in the ordering of elements of sets and not in the sets themselves.

By passing from the category Set of sets to the category Pos of posets, we are able to draw these distinctions with the help of the following order-theoretic notions.

In Pos, concatenation or *ordinal sum* $X; Y$ is definable as cardinal sum (coproduct or juxtaposition) $X + Y$ with its order augmented with $x \leqslant y$ for all $x \in X$, $y \in Y$. Similarly, lexicographic or *ordinal product* $X \cdot Y$ is definable as ordinary or cardinal product $X \times Y$ with its order augmented with $(x, y) \leqslant (x', y')$ for all x, x' in X and all $y < y'$ in Y (so Y supplies the "high-order digit"). Equivalently, $X \cdot Y$ is the result of substituting one copy of X for each element of Y, with the resulting order being that of X within each copy, and that of Y between copies. Just as $A \times X$ can be expressed as the cardinal sum $X + X + \cdots$ of A copies of X, so can $X \cdot A$ be expressed as the ordinal sum $X; X; \dots$ of α copies of X.

Both operations are associative, not commutative, and preserve the posetal extremes of each of linearity (chainhood) and discreteness (sethood), as pointed out by Birkhoff [3, 4], the inventor of the posetal unification of cardinal and ordinal arithmetic.

Birkhoff's colleague Mac Lane at Harvard invented functors too late for any impact on Birkhoff's invention. The sum (coproduct) functor on Set is reflected into Pos to define the extension of $X; Y$ from objects to morphisms: given $f : X \to X'$ and $g : Y \to Y'$, $f; g : X; Y \to X'; Y'$ is defined simply by $(f; g)(x, y) = (f(x), g(y))$, just as for sum, and hence a functor for the same reason (it is clearly monotone). One might suppose the same to be true of ordinal product, given that it is true for cardinal product, but monotonicity fails for the second argument as may be illustrated with the quotient $f : 2 \to 1$ of the two-element chain $2 = \{0 < 1\}$ to the singleton poset $\{0\}$. Consider the morphism $1_2 \cdot f : (2 \cdot 2) \to (2 \cdot 1)$. The four pairs in $2 \cdot 2$ satisfy $00 < 10 < 01 < 11$, while the two pairs in $2 \cdot 1$ are $00 < 10$. Now $2 \cdot f$ must take 00 and 01 to 00, and 10 and 11 to 10. But then we have $(2 \cdot f)(10) > (2 \cdot f)(01)$ despite $10 < 01$, contradicting monotonicity of $2 \cdot f$.

Ordinal product is however functorial in its first argument. To see this, let X vary while holding Y fixed. It suffices to show that for any monotone function $f : X \to X'$ (the agent of variation), the function $f \cdot 1_Y : X \cdot Y \to X' \cdot Y$ given by $(f \cdot 1_Y)(x, y) = (f(x), y)$ is monotone. That is, if $(x, y) \leqslant (x', y')$ then $(f(x), y) \leqslant (f(x'), y')$. But when $y \neq y'$ the order depends only on y and y', while when $y = y'$ it depends only on x and x', satisfying monotonicity in either case.

We also define a functor X^o which replaces X by its order dual. Although this functor reverses the order of elements within a poset, it leaves unchanged the direction and monotonicity of functions between posets, since applying it to f inverts both the domain and codomain of f.

We now separate the two functors $N \times X$ and $N \times X + 1$ into four functors $X \cdot \omega$, $(X \cdot \omega)^o$, $(1; X \cdot \omega)^o$, and $X \cdot \omega; 1$. In this passage we replaced the set N by the ordinal ω, and rendered cardinal sum $X + Y$ as ordinal sum $X; Y$ and cartesian product $X \times Y$

as ordinal product $Y \cdot X$, following ordinal arithmetic's "little-endian" convention of putting the higher order "digits" later.

Theorem 5.1. *The following functors have the associated final coalgebras.*
  $X \cdot \omega : \omega^{\omega}$ *lexicographically ordered, i.e. with the order of* $\ldots \cdot \omega \cdot \omega \cdot \omega$.
  $(X \cdot \omega)^{\mathrm{o}} : \omega^{\omega}$ *with odd-alternating lexicographic order* (*sequences first differing in an even-index* (*resp. odd-index*) *position* (*starting with index* 0) *have the opposite* (*resp. same*) *order as their differing digits*).
  $(1; X \cdot \omega)^{\mathrm{o}} : \omega^{*} \cup \omega^{\omega}$ *with odd-alternating lexicographic order.*
  $X \cdot \omega; 1 : \omega^{*} \cup \omega^{\omega}$ *lexicographically ordered.*

We leave these as straightforward exercises.

Theorem 5.2. *The order types of the final coalgebras of the preceding theorem are respectively the continuum, Baire space, the continuum again, and Cantor space.*

Proof. This follows by essentially the same arguments as for Theorems 3.1–3.4. Each theorem features a final coalgebra Ω for the associated signature. We show as before, ignoring order, that $\mathscr{A}$, $\mathscr{B}$, $\mathscr{A}'$, and $\mathscr{C}$ are isomorphic to the "standard" final coalgebra of their respective associated functors. This makes use of the fact that all underlying sets are as before, whence the bijections are all as before. The final step is to verify that all functions encountered in those proofs are monotone, requiring no creativity but only patient checking. □

Since final algebras are determined up to isomorphism, which in the category of posets means order isomorphism, these order types are uniquely determined as the final coalgebras of the corresponding functors. Thus for all three of the half-open continuum, Baire space, and Cantor space, we have been able to precisely specify their order types as the final coalgebra of the appropriate functor, with two choices of functor in the case of the continuum.

Since the topology of these spaces is determined by their order, being the induced order topology [24] in all cases, it follows that we have also specified the topologies of these three spaces by final coalgebras.

6. Reflections and conclusion

The general result: The basis for the continued fraction expansion of reals is the hyperbola $1/x$, suitable variants of which yield a bijection between $[0, \infty)$ and either $[0, 1)$ or $(0, 1]$, monotone and antimonotone, respectively.

However these are far from the only bijections between these sets of reals, as we have already seen with our variant of Hausdorff's example. We did not consider that example in as much detail as the continued fraction expansions, in particular omitting the explicit specification of its coalgebraic structure. The reader may verify that its

destructors are $h(x) = \lfloor \log_2(1/(1-x)) \rfloor$ and $t(x) = x2^{1+h(x)} \bmod 1$. The choice of base 2 for the logarithm reflects the use of binary expansion; any other radix greater than unity, with the 2 in the definition of t also replaced by that radix, would yield another final coalgebraic structure on $[0,1)$. Since even a non-integer radix will work here while retaining the property that the sequences consist of nonnegative integers, we obtain a continuum of final coalgebra structures.

The tangent function constitutes a familiar monotone bijection between $[0, \pi/2)$ and $[0, \infty)$. Hence taking $h(x) = \lfloor y \rfloor$, $t(x) = y \bmod 1$ where $y = \tan(\pi x/2)$ provides yet another coalgebra $X \to N \times X$ for the continuum. By analogy with the continued fraction and binary expansions, one could call this the circular expansion of the reals. The circular expansion of $1/e$ starts out $0, 1, 1, 1, 0, 1, 1, 2, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 14, 0, 0, 0, 2, 6, 0, 0, 1, 0, 0, \ldots$. The preponderance of 0's here is typical of the circular expansion.

In complete generality (using only the techniques of this paper), any monotone or antimonotone bijection between respectively $[0, 1)$ or $(0, 1]$ and $[0, \infty)$ gives rise to a final coalgebra $[0, 1) \to N \times [0, 1)$ or $(0, 1] \to 1 + N \times (0, 1]$ respectively, and via the bijection a final coalgebra $[0, \infty) \to N \times [0, \infty)$ or $[0, \infty) \to 1 + N \times [0, \infty)$ respectively. For every such final coalgebra, ordering the associated sequences, respectively, lexicographically or alternating lexicographically yields the order type of the continuum.

Paradoxical aspects: Every looking-glass world has its paradoxes, and the continuum viewed as a coalgebra has a couple of striking ones, rendering the continuous discontinuous and vice versa.

With respect to the order topology of the continuum, real addition and multiplication are continuous in each argument separately, and even jointly ($x + x$ and xx are continuous in x). With respect to any coalgebraic structure of the continuum however, addition and multiplication are not "coinductively continuous" in the sense that a finite initial segment of the expansion of either $x + y$ or xy need not be determined by any finite initial segment of the expansions of x and y where x is irrational and y is $1 - x$ or $1/x$, respectively.

Carry propagation for either the binary or decimal expansion of reals provides a familiar case of this. When $x + y$ is a power of the radix, we cannot deduce even the first digit of $x + y$ from any finite initial segment of the binary expansions of x and y (consider the binary expansions $0.01010101\ldots + 0.001010101\ldots$). The underlying discontinuity here is that created by the cut in the continuum defined by the choice of first digit. This discontinuity survives the passage to the Hausdorff coalgebra counting runs of 1's: as long as we cannot tell where the first 1 occurs we certainly cannot determine the first integer in the sequence.

But radix arithmetic is not special in this respect. All final coalgebras $[0, \infty) \to N \times [0, \infty)$ or $[0, \infty) \to 1 + N \times [0, \infty)$ must have essentially the same discontinuities, since for either of these two functors, the final coalgebras of that functor are all order isomorphic and hence homeomorphic under the interval topology, and addition and multiplication are continuous.

To make this point less awkwardly it helps to have the domain be closed under addition, which the unit interval is not. Now the common transformation underlying

all the coalgebraic structures of the previous section is some monotone or antimonotone bijection stretching the unit interval to $[0, \infty)$. Whatever coalgebraic structure we assign to the unit interval can therefore also be taken as coalgebraic structure for $[0, \infty)$ via this bijection. For all coalgebras treated above, this can be understood as equivalent to prepending the integer part of x (suitably defined for the case at hand) to the sequence previously used to represent its fractional part (again suitably defined).

For example the coalgebraic structure of $(0, 1]$ determined by the standard continued fraction expansion of Theorem 3.3, having odd-alternating lexicographic order (monotone in A_{2i+1}, antimonotone in A_{2i}) on finite and infinite sequences, transfers to $[0, \infty)$ via the antimonotone bijection $(1-x)/x$, in the process changing the lexicographic order to even-alternating. This means that the sequence that previously denoted $x \in (0, 1]$ now denotes the element $(1 - x)/x$ of $[0, \infty)$. In this case this is equivalent to prepending $A_0 = \lceil x \rceil - 1 \, (= \lfloor x \rfloor$ for non-integer x but $x - 1$ for integer x, the notion of "integer part" appropriate here) to the sequence previously denoting $x - A_0$, the fractional residue. For example the sequence $(2, 1)$ previously denoted $1/(1+2+1/(1+1)) = \frac{2}{7}$, so the sequence $(3, 2, 1)$ now denotes $3\frac{2}{7}$ (and previously denoted $1/(1+3+1/(1+2+1/(1+1))) = \frac{7}{30}$). The unit length sequence (i) previously denoted $1/(i + 2)$ and now denotes the integer $i + 1$ while the empty sequence previously denoted 1 and now denotes 0.

We may now address addition on $[0, \infty)$ for this coalgebra. The infinite sequences $(0, 0, 0, 0, \ldots)$ and $(0, 2, 0, 0, 0, 0, \ldots)$ now denote respectively $(\sqrt{5}-1)/2 = 0.6180\ldots$ and $(3 - \sqrt{5})/2 = 0.3819\ldots$, which sum to 1, now denoted by the one-element sequence (0). But if we were to change A_{1000} or $A_{10^{100}}$ of either sequence to 1, the sum would increase very slightly (even-alternating lexicographic order being monotone in the even positions and antimonotone in the odd) and its representing sequence would then start with 1. Thus we have the same dependence of the first digit of the sum on the whole of the two summands as with radix notation.

The converse paradox is the constructivity of the (necessarily discontinuous) bijections that can be constructed between the continuum and either Baire space or Cantor space by suitably reinterpreting sequences. These sequences are essentially the only constructive access we have to any of these spaces, and can be understood as the basis for all computationally practical approaches to infinite precision arithmetic. Furthermore none of those considered explicitly in this paper is particularly more constructive than any other. Hence we can use them to constructively pair up reals in the continuum $[0, 1)$ with the irrationals in Baire space $\underline{(0, 1)}$ via the bijections of each with N^N given by Theorems 3.1 and 3.2, respectively. Similarly we can constructively pair up reals in the continuum $(0, 1]$ with the elements of the Cantor space $\overline{[0, 1]}$. The bijection $1 - x$ between $[0, 1)$ and $(0, 1]$ then yields by composition a further constructive bijection between Baire space and Cantor space.

These computable bijections are evidently wildly discontinuous, directly contradicting the intuitions of a generation of programming language semanticists concerning the relationship between computability and continuity. The "coinductive discontinuity" of addition, while not normally considered paradoxical, is at least a familiar phenomenon, whereas a constructive bijection between such wildly non-homeomorphic spaces as

Baire space and the continuum has not to our knowledge even been previously contemplated, let alone considered a paradox.

A side application: The well-known equicardinality of the real line and the plane has an ostensibly simple description in terms of perfectly shuffling the bits of the binary expansions of the coordinates of the points in the plane to yield a single bit stream representing a single real. This is not quite a bijection however, as the real $\frac{1}{3} = 0.010101\ldots$ would have to arise as the perfect shuffle of 0 with $0.1111\ldots$ but the latter is not a legal binary expansion.

The representation of non-negative reals by N^N however has no such exceptions, and points $x \geqslant 0$ in the right half of the real line can be paired bijectively with points (x, y), $x, y \geqslant 0$, in the upper right quadrant by perfectly shuffling the two sequences denoting x and y. The computability of this shuffle is made clear by its simple coinductive definition:

$$h(x \| y) = h(x),$$
$$t(x \| y) = y \| t(x).$$

In the same sense that the bijections of the preceding section are computable, so is this bijection between the real line and the plane. It is necessarily discontinuous, and would ordinarily be considered wildly non-computable. Yet coinductively speaking it is more computable than addition since unlike addition, perfect shuffle never has to look infinitely far ahead to output the next term.

Note that the other coalgebra we studied for the continuum, based on alternating lexicographic ordering, does not help here because to give the continuum it must be applied to $N^* \cup N^N$, but then we have sequences of mixed lengths, which cannot be shuffled perfectly. With continued fractions therefore, this technique only works for the less standard kind treated in Theorem 3.1.

These bijections between the line and the square can be easily turned into bijections between Baire space and its powers, and between Cantor space and its powers, simply by using the bijections we have obtained between the continuum, Baire space, and Cantor space.

Coinduction as induction on predicates: It is reasonable to ask why the continuum should be defined coinductively rather than inductively. It seems to us that the coinductive nature of the continuum is a consequence of our computing not with the reals themselves but inductively with rationals as approximations to reals. The computational status of the reals is not as elements in their own right but as predicates on the rationals, specifically the Dedekind cuts in the rationals. But elements and predicates are dual notions: whereas elements of a set X can be understand as functions *to* X, predicates on X are functions *from* X.[6] Induction performed on the structures underlying the rationals therefore looks like coinduction when we view it as though applied to the reals themselves.

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[6] Normally the functions to X denoting elements are from a singleton while those from X denoting predicates are to a doubleton, but a good deal of the structure of mathematics is insensitive to these choices.

Alternative definitions of the continuum: It is worth mentioning a few of the other ways the continuum has been defined for comparison.

Best known of course is Dedekind's original definition as cuts in the rational line. Here one is confronted with a variety of definitions of "cut". The most obvious, order filters (complementary to order ideals and hence "the same definition"), produces not the continuum but Cantor space as it duplicates rationals. The continuum can be obtained either as a subset by omitting one duplicate of each rational (itself involving the choice of which duplicate to omit), or as a quotient by identifying the duplicates. These choices correspond to distinct perspectives on the relationship between Cantor space and the continuum: the former makes Cantor space the continuum damaged by "extra dirt" while the latter makes it the continuum damaged by seemingly removing open intervals to create gaps with a rational at each end of the gap, in the manner of Cantor's middle-third construction of his space. The field structure on the continuum is inferred by continuity from that of the rationals. At no stage in Dedekind's construction does Baire space enter.

The arbitrariness of these choices may be eliminated in the spirit of Brouwer by allowing a cut to determine three possibilities for a rational rather than the customary two. Define a real to be a "predicate" on the rationals with truth values L, M, R for left, middle and right, such that $L < M < R$ ($L(x) \wedge M(y) \rightarrow x < y$ etc.), L and R have no endpoint ($L(x) \rightarrow \exists y > x \cdot L(y)$ and dually for R), and $|M| \leqslant 1$ ($M(x) \wedge M(y) \rightarrow x = y$). A real is then rational just when M is non-empty.

A definition less well-known if only because of its greater sophistication directly yields the field of reals from the ring of Cauchy sequences of rationals, where the ring operations are defined pointwise. The field of reals is obtained as the quotient of the ring R of Cauchy sequences by the ideal of R generated by the vanishing sequences.

Conway's remarkable notion of a game as a pair of sets of numbers [5] forms the basis of an inductive definition of number starting with just the empty set as the number 0. A game is a number just when it is a cut, i.e. every number in one set is less than every number in the other. In this way Conway obtains first 0, then 1 and its negation, then $\frac{1}{2}$, 2, and their negations, then $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{2}$, 3, and their negations, and so on. In this way he obtains all "binary rationals", which unlike the rationals however do not form a field ($\frac{1}{3} = 0.010101\ldots$ in binary repeats infinitely and so is not a binary rational).

But then by continuing the construction at infinite ordinals Conway obtains all the remaining rationals immediately (i.e. on "day" ω), yet still has no field as the infinite number ω and the infinitesimal $1/\omega$ also enter on day ω but without $\omega + 1$ or $2/\omega$, which do not enter until day $\omega + 1$. By continuing the construction up through all the ordinals he obtains a field *No*, or rather a Field as it has grown so large by that point as to constitute a proper class. An earlier stopping point should suffice however to produce a field, one well endowed with algebraically robust infinities and infinitesimals, certainly by day ε_0 if not sooner. But even by day ω the presence of ω and $1/\omega$ make it too late to hope for an Archimedean field (one isomorphic to a subfield of the reals; equivalently, an ordered field with no infinite elements; equivalently, no infinitesimals).

The advantage for Conway of continuing "forever" is that *No* is as richly structured as any field, being indeed the universally embedding ordered field, meaning that it contains as subfields up to isomorphism all extensions of its subfields. In particular it must embed Robinson's nonstandard-analysis field.

The upshot of all this is that Conway defines the ultimate extension of the field of reals, but not the field of reals itself.

Yet another definition bypasses the rationals altogether and passes in one step from the additive group of bounded sequences of integers (addition is coordinatewise, and for each sequence $a_0, a_1, \ldots$ there exists an integer m_a such that $\forall i.|a_i| \leqslant m_a$) to the additive group G of reals, thence in a second (standard) step to the field of reals. Let H consist of those sequences $b \in G$ for which there exists a sequence $a \in G$ such that $b_i = a_i - 2a_{i+1}$ for all i. H is easily seen to be a normal subgroup of G, and G/H can then be shown by induction on the bounds m_a to be isomorphic to the additive group of reals. (The effect of this quotient is to identify every sequence with a sequence all but whose leading element is either 0 or 1, and furthermore having no infinite run of 1's, in effect an integer followed by the binary expansion of a real in $[0, 1)$, which when added to the integer gives the so-represented real.) The group G/H of reals then expands to the ring (hence field) of reals by standard means (take the endomorphisms of G/H to be the scalar multiplications and then unify these into one binary multiplication by pairing the identity endomorphism with the real 1).

Knuth and Pratt [13] combine the above two steps into one by weakening the $O(1)$ bound on a_i to $O((\frac{3}{2})^i / i^{3/2})$, which allows multiplication to be defined on the resulting group as convolution to make it a ring R, equivalent to taking sequences to be formal power series. The subset of R defined as for H above is an ideal I of R, and R/I is then obtained as the ring (hence field) of reals in a single step starting with this ring of integer sequences.

Common to all these definitions is the immediate goal of specifying ordered field structure. In contrast the definition of this paper focuses instead on coalgebraic structure, from which all remaining structure is then defined coinductively, albeit not necessarily "continuously" with respect to that structure as with sum and product.

Conclusions: We showed that the final coalgebra structure for infinite sequences over N, the nonnegative integers, permits coinductive definitions of the order type of both the continuum and Baire space, restricted to the unit interval, by ordering the sequences in either lexicographic or alternating lexicographic order, respectively. We also showed that if finite sequences are allowed as well as infinite, the corresponding orderings give the order type of Cantor space and the continuum respectively. The standard topology on each of these spaces can then be obtained in turn as the order topology induced by the respective orders.

The representation of the continuum in terms of sequences is well known, perhaps best via the continued fraction expansion. What has not been previously pointed out is that this makes possible a coinductive definition of the continuum, and for that matter Baire space and Cantor space, putting all three of these spaces on a more equal footing with the inductive definition of the natural numbers.

We close with the following questions:

1. What satisfactory resolutions of the above-mentioned paradoxes are there? Does there exist for example a topology making functions such as perfect shuffle having a "smooth" coinductive definition continuous while making "rough" ones like addition discontinuous? Would such a topology if it existed be dual in some direct sense to the standard topology on the continuum?

2. Do the coinductive definitions of Baire space and Cantor space make them any more useful in practice? Would there be any point in having a datatype of either, as an adjunct to a datatype of infinite-precision reals?

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